Positive solutions for boundary value problem of fractional differential equations with the first order derivatives

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Abstract: In this paper, by constructing the appropriate operator, using a new cone fixed point theorem. The existence of at least one positive solutions for boundary value problem of fractional differential equations with the first order derivative

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0, 0<t<1; \\ u(0)=u'(1)=u''(0)=0 \end{cases}$$

is considered, where f with the first derivative. $2<\alpha<3$, $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$.

1. Introduction

Recently， With the development of nonlinear science， The researchers found that using fractional differential equations can more accurately describe the changing laws of natural phenomena. where， Fractional differential equations have important applications in cybernetics, hydrology, chemistry, signal recognition and other fields， His research has attracted wide attention and some related monographs have been published[1]. therefore， It is very important to study the boundary value problems of fractional differential equations for solving nonlinear problems[7-8,10].

Bai [2] investigated the existence of positive solutions for fractional differential equations, by the use of a fixed point theorem.

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1; \\ u(0)=u(1)=0 \end{cases}$$

Where, $1<\alpha\le 2$, f is continuous, by a fixed point theorem in cone, the existence of at least one positive solutions of BVP .

Kosmatov [3] investigated the existence of positive solutions for fractional differential equations.

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0, 0<t<1; \\ u(0)=u(1)=0 \end{cases}$$

where $1<\alpha\le 2$ is a real number, D_{0+}^{α} is a $R-L$ fractional derivative of standard, function $f\in L'[0,1]$ is satisfying condition of $Carath\tilde{e}$.

Liu [4] investigated the existence of Multiple positive solutions for fractional differential equations.

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0+}^{\alpha}u(t))=0, 0<t<1; \\ u(0)=u'(1)=u''(0)=0 \end{cases}$$

where $2<\alpha<3, 1<\beta\le 2$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous functional.

All the above works ,In this paper, we are concerned with the existence of positive solutions for boundary value problem of fractional differential equations with the first order derivatives

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$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0, 0<t<1; \\ u(0)=u'(1)=u''(0)=0 \end{cases} \quad (1)$$

where $2<\alpha<3, 1<\beta\le 2$, $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous functional.

2. The preliminary lemmas

Define 2.1[5] Function $y:(0,+\infty)\to R$ is integral of Riemann-Liouville, $\alpha>0$, then

$$I_{0+}^{\alpha}y(t)=\frac{1}{P(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

where The right hand side is defined in $(0,+\infty)$.

Define 2.2[6] Function $y:(0,+\infty)\to R$ is fractional derivative of Caputo, $\alpha>0$, then

$$D_{0+}^{\alpha}y(t)=\frac{1}{p(m-\alpha)}\int_0^t \frac{y^m(s)}{(t-s)^{\alpha-m+1}}ds$$

where $0\le m-1\le\alpha<m$.

Lemma 2.1[6] Let $\alpha>0$, fractional differential equations $D_{0+}^{\alpha}u(t)=0$ has the unique solution:

$$u(t)=c_0+c_1t+c_2t^2+\cdots+c_it^i+\cdots+c_{n-1}t^{n-1}, \quad c_i\in R, i=0,1,2,\cdots,N$$

where $N=[\alpha]+1$.

Lemma 2.2[9] （Schauder fixed point theorem） Let E is a Banach space, K is a bounded closed convex subset of E. $T:K\to K$ Is a completely continuous operator, then T have a fixed point in P.

Lemma 2.3[9] Let $r_2>r_1>0, L>0$ be constants and

$\Omega_i=\{u\in X:\sigma(u)<r_i,\tau(u)<L\}, i=1,2$, two bounded open sets in X,

Set $D_i=\{u\in X:\sigma(u)=r_i\}, i=1,2$ .Assume $T:K\to K$ is a completely continuous operator satisfying:

（A$_1$） $\sigma(Tu)<r_1, u\in D_1\cap K; \sigma(Tu)>r_2, u\in D_2\cap K$;

（A$_2$） $\tau(Tu)<L, u\in K$;

（A$_3$） there is $p\in(\Omega_2\cap K)\setminus\{0\}$, such that $\sigma(p)\ne 0$ and $\sigma(u+\lambda p)\ge\sigma(u)$ for all $u\in K, \lambda\ge 0$.

Then T has at least one fixed point in $(\Omega_2\setminus\bar{\Omega}_1)\cap K$.

Lemma 2.4[4] For any $f(t)\in C(0,1), 2<\alpha\le 3$, then

$$\begin{cases} -D_{0+}^{\alpha}u(t)=f(t), t\in[0,1]; \\ u(0)=u'(1)=u''(0)=0 \end{cases} \quad (2)$$

has the unique solution : $u(t)=\int_0^1 G(t,s)f(s)ds$ （3）

$$\text{where } G(t,s)=\begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, 0\le s\le t\le 1 \\ \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \qquad 0\le t\le s\le 1 \end{cases} \quad (4)$$

By（3）,we have $u'(t)=\int_0^1 \frac{\partial G(t,s)}{\partial t}f(s)ds$.

Lemma 2.5[4] For any $f(t)\in C(0,1)$,then $G(t,s)$ has:

（1） $G(t,s)>0$, where $G(t,s)\in C([0,1]\times[0,1]), t,s\in(0,1)$;

（2） $\max\limits_{0\le t\le 1} G(t,s)=G(1,s), s\in[0,1]$;

（3） $\exists r(s)\in C(0,1), s\in(0,1)$, and satisfying:

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s)\ge r(s)M(s) \qquad (5)$$

$$\max_{0\le t\le 1} G(t,s)\le M(s) \qquad (6)$$

where, $M(s)=\dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$, $r(s)=\dfrac{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}{4(\alpha-1)(1-s)^{\alpha-2}}$.

3. The main results

Let X be the Banach space, $K\subset X$ is a cone. Suppose $\alpha,\beta: X\to R^{+}$ are two convex functionas, satisfying: $\alpha(\lambda u)=|\lambda|\alpha(u), \beta(\lambda u)=|\lambda|\beta(u),\ u\in X, \lambda\in R$.

and $\|u\|\le M\max\{\alpha(u),\beta(u)\}$, $u\in X$; $\alpha(u)\le\alpha(v)$, $u,v\in K, u\le v$

where $M>0$ be constant.

Define Banach space: $X=\{u|u(t)\in C^{1}[0,1]\}$ equipped with the norm $\|u\|=\max\limits_{t\in[0,1]}|u(t)|, \|u'\|=\max\limits_{t\in[0,1]}|u'(t)|$, K is a cone in X , $K=\{u\in X, u(t)\ge 0\}$.

Define functionals $\alpha(u)=\max\limits_{t\in[0,1]}|u(t)|, \beta(u)=\max\limits_{t\in[0,1]}|u'(t)|$, $\forall u\in X$; then $\|u\|\le 2\max\{\alpha(u),\beta(u)\}$, $\alpha(\lambda u)=|\lambda|\alpha(u), \beta(\lambda u)=|\lambda|\beta(u)$, $\forall u\in X, \lambda\in R$; $\alpha(u)\le\alpha(v)$, $u,v\in K, u\le v$.

In the following, we denote : $M=\int_0^1 M(s)ds$, $N=\int_{\frac{1}{4}}^{\frac{3}{4}} r(s)M(s)ds$, $Q=\dfrac{2}{\Gamma(\alpha)}$

We will suppose that there are $L>b>rb>c>0$, such that $f(t,u,v)$ satisfies the following growth conditions:

（H_1） $f(t,u,v)<\dfrac{c}{M}$, $(t,u,v)\in[0,1]\times[0,c]\times[-L,L]$

（H_2） $f(t,u,v)\ge\dfrac{b}{N}$, $(t,u,v)\in[\dfrac{1}{4},\dfrac{3}{4}]\times[rb,b]\times[-L,L]$

（H_3） $f(t,u,v)<\dfrac{L}{Q}$, $(t,u,v)\in[0,1]\times[0,b]\times[-L,L]$

Let $\tilde{f}(t,u,v)=\begin{cases} f(t,u,v), (t,u,v)\in[0,1]\times[0,b]\times(-\infty,+\infty) \\ f(t,u,v), (t,u,v)\in[0,1]\times[b,+\infty)\times(-\infty,+\infty)\end{cases}$

and $\overline{f}(t,u,v)=\begin{cases} \tilde{f}(t,u,-L), (t,u,v)\in[0,1]\times[0,+\infty)\times(-\infty,-L] \\ \tilde{f}(t,u,v), (t,u,v)\in[0,1]\times[0,+\infty\times[-L,L] \\ \tilde{f}(t,u,L), (t,u,v)\in[0,1]\times[0,+\infty)\times[L,+\infty)\end{cases}$

Define :

$$\begin{aligned}(Tu)(t)&=\int_0^1 G(t,s)\overline{f}(s,u,v)ds \\ &=\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-2}}{\Gamma(\alpha)}\overline{f}(s,u,v)ds+\int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}\overline{f}(s,u,v)ds \\ &=\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\overline{f}(s,u,v)ds-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\overline{f}(s,u,v)ds\end{aligned}$$

$$(Tu)'(t)=\int_0^1 \frac{\partial G(t,s)}{\partial t}\,\overline{f}(s,u,v)ds$$

$$=\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\overline{f}(s,u,v)ds-\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\overline{f}(s,u,v)ds$$

Lemma 3.1: Suppose $f(s,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty)$ is continuous, then $T:K\to K$ is completely continuous.

Proof: By define of $(Tu)(t)$, there is $Tu\in C[0,1]\times C^1[0,1]$ and $Tu\ge 0$ for all $\forall u\in K$. we have $T(K)\subset K$.

For $f(s,u,v)\in C([0,1]\times[0,+\infty)\times[0,+\infty))$ and $G(t,s)\in C([0,1]\times[0,1])$, we have T is continuous.

Under T is a compact operator, let $\Omega\subset K$ is bound,then $\exists a>0$, such that $\|u\|\le a,\|u'\|\le a,u\in\Omega$.

Let $\theta=\max\limits_{0\le u\le a,0\le u'\le a}\left|f(s,u,u')\right|+1$, then $\forall u\in\Omega$, by Lemma 2.3,there is

$$\left|(Tu)(t)\right|=\left|\int_0^1 G(t,s)f(s,u,u')ds\right|\le\int_0^1\left|G(t,s)f(s,u,u')\right|ds$$

$$\le\theta\int_0^1 M(s)ds=\theta\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds$$

$$=\frac{\theta}{\Gamma(\alpha)}\le\frac{2\theta}{\Gamma(\alpha)}$$

$$\left|(Tu)'(t)\right|=\left|\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u,u')ds-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u,u')ds\right|$$

$$\le\theta(\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds)$$

$$=\frac{\theta}{\Gamma(\alpha)}+\frac{\theta t^{\alpha-1}}{\Gamma(\alpha)}\le\frac{2\theta}{\Gamma(\alpha)}$$

By Lemma 2.5 $t\in[0,1]$, $\left|(Tu)'(t)\right|\le\theta(\frac{1}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)})\le\frac{2\theta}{\Gamma(\alpha)}$, so $T(K)$ uniform bound.

On the other hand, let $t_1,t_2\in[0,1],t_1<t_2,u\in\Omega$,then have

$$\left|(Tu)(t_2)-(Tu)(t_1)\right|=\left|\int_0^1 G(t_2,s)f(s,u,u')ds-\int_0^1 G(t_1,s)f(s,u,u')ds\right|$$

$$=\left|\int_0^1\frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u,u')ds-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,u')ds+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u,u')ds\right|$$

$$\le\theta\int_0^1\left|\frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right|ds+\theta\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|$$

$$\le\frac{\theta}{\Gamma(\alpha)}(t_2-t_1)+\frac{\theta}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})$$

$$\left|(Tu)'(t_2)-(Tu)'(t_1)\right|=\left|\int_0^1\frac{\partial G(t_2,s)}{\partial t_2}f(s,u,u')ds-\int_0^1\frac{\partial G(t_1,s)}{\partial t_1}f(s,u,u')ds\right|$$

$$=\frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_1}(t_1-s)^{\alpha-2}f(s,u,u')ds-\int_0^{t_2}(t_2-s)^{\alpha-2}f(s,u,u')ds\right|$$

$$=\frac{\theta}{\Gamma(\alpha)}[t_2^{\alpha-1}-t_1^{\alpha-1}]$$

In conclusion, T is completely continuous.

Theorem 3.1: For $f(s,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty)$ is continuous, and condition（H_1）-（H_3）hold, Then BVP（1）has at least one positive solution $u(t)$, satisfying

$$c<\alpha(u)<b,\beta(u)<L$$

Proof : Take $\Omega_1=\{u\in X:\alpha(u)<c,\beta(u)<L\}$, $\Omega_2=\{u\in X:\alpha(u)<b,\beta(u)<L\}$ two bounded open sets in X; and $D_1=\{u\in X:\alpha(u)=c\}$, $D_2=\{u\in X:\alpha(u)=b\}$.

By Lemma 3.1, $T:K\to K$ is completely continuous operator, and there is $\exists p\in(\Omega_2\cap K)\setminus\{0\},\alpha(p)\neq 0$, such that $\alpha(u+\lambda p)\geq\alpha(u)$, where $u\in K,\lambda\geq 0$.

（1）By（H_1）and $\alpha(u)=c,u\in D_1\cap K$, we have

$$\alpha(Tu)=\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\overline{f}(t,u,v)ds\right|<\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\frac{c}{M}ds\right|\leq\frac{c}{M}\int_0^1 M(s)ds=c$$

（2）$u(t)\geq r\|u\|,t\in[\frac{1}{4},\frac{3}{4}]$, for $\alpha(u)=b,u\in D_2\cap K$ and（H_2）, we get

$$\alpha(Tu)=\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\overline{f}(t,u,v)ds\right|>\max_{t\in[\frac{1}{4},\frac{3}{4}]}\left|\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\frac{b}{N}ds\right|>\frac{b}{N}\int_{\frac{1}{4}}^{\frac{3}{4}} r(s)M(s)ds=b$$

（3）By $u\in D_2\cap K$ and（H_3）, we get

$$\beta(Tu)=\max_{t\in[0,1]}\left|\int_0^1\frac{\partial G(t,s)}{\partial t}\overline{f}(t,u,u')ds\right|=\max_{t\in[0,1]}\left|\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u,u')ds-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u,u')ds\right|$$

$$<\frac{1}{\Gamma(\alpha-1)}[\int_0^1(1-s)^{\alpha-2}ds-\int_0^t(t-s)^{\alpha-2}ds]\times\frac{L}{Q}$$

$$=[\frac{1}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}]\times\frac{L}{Q}<\frac{2}{\Gamma(\alpha)}\times\frac{L}{Q}=L$$

Theorem 3.1 implies there is $u\in(\Omega_2\setminus\overline{\Omega}_1)\cap K$, such that $u=Tu$. So $u(t)$ is a positive solution for BVP(1), satisfying :

$$c<\alpha(u)<b,\beta(u)<L$$

4. Conclusion

In this paper, the existence of at least one solutions to boundary value problem of fractional differential equations with one order derivative is considered; by constructing the appropriate operator, using a new cone fixed point theorem. the existence of solution is verified .

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References

[1] Killbas A A, Srivastava H M. Trujillo J J. Theory and Application of Application of Fraction Differential Equation, North-Holland Mathematic Studies 204[J]. *Amsterdam: Elsevier Science B V*, 2006.

[2] Bai Z, L ü H. Positive solutions for boundary value problem of nonlinear fractional differential equation [J]. *Journal of Mathematical Analysis and Applications*, 2005, 311(2): 495-505.

[3] Kosmatov N. A singular boundary value problem for nonlinear differential equations of fractional order [J]. *Journal of Applied Mathematics and Computing*, 2009, 29(1-2): 125-135.

[4] Yang Liu, Zhi hui Liu, Cheng zhi Li, Wei min Hu, The existence of multiple positive solutions for T boundary value problems of fractional-order differential equations with fractional derivative

terms. *Mathematics in Practice and Theory* 2016，46（4）:242-248.
[5] Wang G, Ntouyas S K, Zhang L. Positive solutions of the three-point boundary value problem for fractional-order differential equations with an advanced argument [J]. *Advances in Difference Equations*, 2011, 2011(1):348-357.
[6] Chai G. Positive solutions for boundary value problem of fractional differential equation with p-Laplace operator [J]. *Boundary Value Problems*, 2012, 2012(1):1-20.
[7] Yang X, Wei Z, Dong W. Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations [J]. *Communications in Nonlinear Science and Numerical Simulation*, 2012, 17(1):85-92.
[8] Bai Z, L ü H. Positive solutions for boundary value problem of nonlinear fractional differential equation[J]. *Journal of Mathematical Analysis and Applications*, 2005, 311(2): 495-505.
[9] Ji Y, Guo Y, Zhang J. Positive solutions for second-order quasilinear multi-point boundary value problem [J]. *Applied Mathematical and computation*, 2007, 189(1), 725-731.
[10] Feng MQ, Zhang X M and Ge WG et al. New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions [J]. *BVP-Boundary Value Problems*, 2011, 2011(10): 1-20.